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NECESSARY AND SUFFICIENT CONDITIONS THAT AN ORDINARY DIFFERENTIAL EQUATION SHALL ADMIT A CONFORMAL GROUP

By L. I. Hewes

1. Introduction. We are to study in this paper a necessary and sufficient condition that an ordinary differential equation of the first order

$$\frac{dy}{dx} = a(x,y) \tag{1}$$

shall admit a continuous one-parameter conformal group. We shall assume the general properties of conformal transformation groups and in particular the following theorem due to Professor Bouton: [*]

A one-parameter group of conformal transformations with given path curves exists when and only when the given curves form an isothermal family.

We also assume the general methods laid down by Lie [†] and used by Tresse [‡] in the study of differential invariants.

It will be shown that under the infinite group of all conformal transformations there exist differential invariants formed from $a(x, y)$ and its partial derivatives. There are two of the third order and none of lower order. An infinite number of higher order are readily obtained from these two of lowest order. The desired condition for admission to occur is found to be dependent upon the nature of the invariants of the third and fourth orders.

2. The Differential Invariants. The discussion is greatly simplified by the introduction of circular coordinates [§]

$$u = x + iy, \quad v = x - iy, \qquad i = \sqrt{-1}.$$

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[*] *Bull. Am. Math. Soc.* vol. 11, p. 369.

[†] *Math. Ann.*, vol. 24, pp. 550-576.

[‡] *Acta Math.* vol. 18, p. 76, also *Preisshrift, Jab. Gesell. zu Leipzig*, 1893.

[§] F. Franklin, *Am. Journ. Math.*, vol. 12, 1890, p. 161.

The finite transformations of the infinite conformal group are then

$$u' = \Phi(u), \qquad v' = \Psi(v),$$

and the corresponding infinitesimal symbol may be written:

$$Uf: \qquad \phi(u)\frac{\partial f}{\partial u} + \psi(v)\frac{\partial f}{\partial v},$$

where $\phi(u)$ and $\psi(v)$ denote the first derivatives of $\Phi(u)$ and $\Psi(v)$ respectively. If we extend Uf to operate upon $\frac{dv}{du}$ we have

$$U'f: \qquad \phi(u)\frac{\partial f}{\partial u} + \psi(v)\frac{\partial f}{\partial v} + \pi(\psi'(v) - \phi'(u))\frac{\partial f}{\partial \pi},$$

when π denotes $\frac{dv}{du}$. In the new coordinates the differential equation (1) becomes

$$(2) \qquad \frac{dv}{du} = m(u, v)$$

or more compactly

$$(2') \qquad \pi - m = 0,$$

and it preserves this form under conformal transformations. If we regard m as a variable transformed by the infinitesimal transformation $U'f$ its increment is determined by the identity

$$m(\psi' - \phi') - \delta m = 0.$$

Lie* has given formulas for computing the increments which the partial derivatives of m will also receive under $U'f$ and these become in the present case

$$\text{(I)} \qquad \frac{\delta m_u}{\delta t} = \frac{\partial}{\partial u}\left(\frac{\delta m}{\delta t}\right) - m_u\phi', \quad \frac{\delta m_v}{\delta t} = \frac{\partial}{\partial v}\left(\frac{\delta m}{\delta t}\right) - m_v\psi',$$

when the subscripts denote partial derivatives.

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* l. c. Also Lie-Engel, vol. 1, p. 545.

By repeated use of these formulas we may compute the following table of increments (δt is omitted) of the successive partial derivatives of m up to the third order.

$$\begin{aligned}
\delta m &= m(\psi' - \phi') \\
\delta m_u &= m_u(\psi' - 2\phi') - m\phi'' \\
\delta m_v &= -\, m_v\phi' + m\psi'' \\
\delta m_{uu} &= m_{uu}\,(\psi' - 3\phi') - 3m_u\phi'' - m\phi''' \\
\delta m_{uv} &= -\, 2m_{uv}\,\phi' + m_u\,\psi'' - m_v\phi'' \\
\delta m_{vv} &= -\, m_{vv}(\phi' + \psi') + m_v\psi'' + m\psi''' \\
\delta m_{uvv} &= -\, m_{uvv}(2\phi' + \psi') + m_{uv}\psi'' - m_{vv}\phi'' + m_u\psi''' \\
\delta m_{uuu} &= m_{uuu}(\psi' - 4\phi') - m_{uu}6\phi'' - 4m_u\phi''' - m\phi'''' \\
\delta m_{uuv} &= -\, 3m_{uuv}\phi' - 3m_{uv}\phi'' + m_{uu}\psi'' - m_v\,\phi''' \\
\delta m_{vvv} &= -\, m_{vvv}(\phi' + 2\psi') + 2m_v\psi''' + m\,\psi''''.
\end{aligned}$$

Then the corresponding infinitesimal operator upon any function F of m and its partial derivatives above is therefore:

$$\mathfrak{U}F \equiv \delta_m\,\frac{\partial F}{\partial m} + \delta m_u\,\frac{\partial F}{\partial m_u} + \cdots + \delta m_{vvv}\,\frac{\partial F}{\partial m_{vvv}}\,.$$

We seek now those functions F of m and its partial derivatives which satisfy the condition:

$$(3) \qquad \mathfrak{U}F = 0.$$

Since this condition is to hold true for all conformal transformations, that is to say, for every ϕ and ψ, we find that the coefficients of ϕ and ψ and their derivatives in equation (3), must vanish. This gives rise to the following complete system of linear partial differential equations:

$$[1]\quad m\frac{\partial F}{\partial m}+2m_u\frac{\partial F}{\partial m_u}+m_v\frac{\partial F}{\partial m_v}+3m_{uu}\frac{\partial F}{\partial m_{uu}}+2m_{uv}\frac{\partial F}{\partial m_{uv}}+m_{vv}\frac{\partial F}{\partial m_v{}^v}$$

$$+4m_{uuu}\frac{\partial F}{\partial m_{uuu}}+3m_{uuv}\frac{\partial F}{\partial m_{uuv}}+2m_{uvv}\frac{\partial F}{\partial m_{uvv}}+m_{vvv}\frac{\partial F}{\partial m_{vvv}}=0$$

$$[2]\quad m\frac{\partial F}{\partial m_u}+3m_u\frac{\partial F}{\partial m_{uu}}+m_v\frac{\partial F}{\partial m_{uv}}+6m_{uu}\frac{\partial F}{\partial m_{uuu}}+3m_{uv}\frac{\partial F}{\partial m_{uuv}}$$

$$+m_{vv}\frac{\partial F}{\partial m_{uvv}}=0$$

$$[3]\quad m\frac{\partial F}{\partial m_{uu}}+4m_u\frac{\partial F}{\partial m_{uuu}}+m_v\frac{\partial F}{\partial m_{uuv}}=0$$

$$[4]\quad m\frac{\partial F}{\partial m_{uuu}}=0$$

$$[5]\quad m\frac{\partial F}{\partial m}+m_u\frac{\partial F}{\partial m_u}+m_{uu}\frac{\partial F}{\partial m_{uu}}-m_{vv}\frac{\partial F}{\partial m_{vv}}+m_{uuu}\frac{\partial F}{\partial m_{uuu}}$$

$$-m_{uvv}\frac{\partial F}{\partial m_{uvv}}-2m_{vvv}\frac{\partial F}{\partial m_{vvv}}=0$$

$$[6]\quad m\frac{\partial F}{\partial m_v}+m_u\frac{\partial F}{\partial m_{uv}}+m_v\frac{\partial F}{\partial m_{vv}}+m_{uu}\frac{\partial F}{\partial m_{uuv}}+m_{uv}\frac{\partial F}{\partial m_{uvv}}=0$$

$$[7]\quad m\frac{\partial F}{\partial m_{vv}}+m_u\frac{\partial F}{\partial m_{uvv}}+2m_v\frac{\partial F}{\partial m_{vvv}}=0$$

$$[8]\quad m\frac{\partial F}{\partial m_{vvv}}=0$$

We thus find that *until we use the derivatives of the third order*, there are not enough variables to give us any solutions. There are plainly two solutions of the system [1] $\cdots$ [8]. To obtain these we adopt as the most practical method the use of the solutions of one equation as new variables which are introduced into each of the remaining equations. This process repeated gives us finally one equation in three variables whose solutions are:

$$X_1=\frac{1}{(m_um_v-mm_{uv})^{\frac{1}{2}}}\left[\frac{m_u}{m^{\frac{1}{2}}}+\frac{1}{3}\,\frac{m^{\frac{1}{2}}(mm_{uuv}-m_vm_{uu})}{(m_um_v-mm_{uv})}\right],$$

$$Y_1=\frac{m^{\frac{1}{2}}}{(m_um_v-mm_{uv})^{\frac{1}{2}}}\left[m_v+\frac{m(mm_{uvv}-m_um_{vv})}{m_um_v-mm_{uv}}\right].$$

These invariants may be given a simpler form as follows:

Let
$$a = \frac{\partial^2}{\partial u \partial v} \log m = \frac{m_{uv}\, m - m_u\, m_v}{m^2}.$$

The function a vanishes identically when and only when the integral curves form an isothermal family. This case is of no interest from the present viewpoint.

Write
$$X = -3iX_1 \quad \text{and} \quad Y = -iY_1,$$
then
$$X = \frac{1}{(ma)^{\frac{1}{2}}} \frac{\partial}{\partial u}\left(\log \frac{a}{m}\right), \quad Y = \left(\frac{m}{a}\right)^{\frac{1}{2}} \frac{\partial}{\partial v}\left(\log am\right).$$

The functions X and Y cannot both vanish identically unless $a \equiv 0$.

Now it is evident that we could calculate the increments of the derivatives of the next higher order viz:

$$\delta m_{uuuu},\ \delta m_{uuuv},\ \delta m_{uuvv},\ \delta m_{uvvv},\ \delta m_{vvvv},$$

and annex the corresponding additional equations to our above system and again seek the solutions. There would be ten equations with fifteen variables and hence three new solutions of the next higher order than X and Y. In general, if the number of invariants of the Kth order is N there will be $N + K$ of the $(K + 1)$st order. We do not, however, need to carry out such a process for we may obtain all the invariants required by a simpler method; namely, that of the "Differential Parameter."

We calculate the increments of the partial derivatives of either invariant, say of

$$X_u \text{ and } X_v\,{}^*$$

under the infinitesimal transformation $U'f$. We find these increments by formulas I:

$$\frac{\delta X_u}{\delta t} = \frac{\partial}{\partial u}\frac{\delta X}{\delta t} - X_u \phi',$$
$$\frac{\delta X_v}{\delta t} = \frac{\partial}{\partial v}\frac{\delta X}{\delta t} - X_v \psi';$$

since X is an invariant function, $\delta X = 0$, and there results:

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* It is assumed that neither X_u nor X_v vanishes identically.

$$\frac{\delta X_u}{\delta t} = -\, X_u \phi', \qquad \frac{\delta X_v}{\delta t} = -\, X_v \psi'.$$

We can now determine if there are invariants $\mathfrak{J}$ which involve

$$m,\ X_u,\ X_v,$$

for they must satisfy the relation

$$-\, X_u \phi' \frac{\partial \mathfrak{J}}{\partial X_u} - X_v \psi' \frac{\partial \mathfrak{J}}{\partial X_v} + m(\psi' - \phi') \frac{\partial \mathfrak{J}}{\partial m} = 0$$

for every ϕ' and ψ'. We have therefore to solve the linear partial differential equations

$$[9] \qquad X_u \frac{\partial \mathfrak{J}}{\partial X_u} + m \frac{\partial \mathfrak{J}}{\partial m} = 0,$$

$$[10] \qquad X_v \frac{\partial \mathfrak{J}}{\partial X_v} - m \frac{\partial \mathfrak{J}}{\partial m} = 0.$$

These equations form a complete system. Assuming that neither X_u nor X_v vanishes identically, a solution is

$$\frac{m X_v{}^{*}}{X_u}.$$

Similarly, using the invariant Y we should obtain the invariant

$$\frac{m Y_v{}^{*}}{Y_u}.$$

We shall call these the first derived invariants and designate them as I and J respectively.

3. The Necessary Condition for Admission.

Consider now the case that the differential equation

$$(2') \qquad \pi - m = 0$$

admits a one-parameter conformal group whose infinitesimal transformation is

$$U'f: \qquad \phi \frac{\partial f}{\partial u} + \psi \frac{\partial f}{\partial v} + \pi(\psi' - \phi') \frac{\partial f}{\partial \pi}.$$

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* The geometric interpretation of these invariants is immediately found to be the tangent of the angle between the integral curves and the curves $X = C$ or $Y = C$ respectively.

By the definition of admission the transformed differential equation is,

$$\pi' - m(u', v') = 0.$$

The general differential invariants X and Y are now invariant functions of u and v as are likewise I and J; they are all therefore as functions of u and v, functions of the unique integral invariant of the group and any one, set equal to a constant, must determine the path curves. They are therefore all functions of the same harmonic function,

$$U + V,$$

where U and V are functions of u and v alone, respectively.

Evidently *all* the differential invariants of higher orders submit to the same condition. We may then state as a necessary condition, which will be shown to be sufficient, the following:

A necessary condition that a first order ordinary differential equation shall admit a one-parameter conformal group is that the invariant X (or Y) and its first derived invariant I (or J) shall be functions of the same harmonic function.

4. The Sufficient Condition. Then $X = c$ determines the unique conformal group:

$$\overline{U}'f: \quad \rho \frac{\partial f}{\partial u} - \rho \frac{X_u}{X_v} \frac{\partial f}{\partial v} - \pi \left(\frac{\partial}{\partial v} \rho \frac{X_u}{X_v} + \frac{\partial}{\partial u} \rho \right) \frac{\partial f}{\partial \pi},$$

where ρ is a function of u and v satisfying the conditions

$$[4] \qquad \frac{\partial \rho}{\partial v} = 0, \qquad \frac{\partial}{\partial u} \rho \frac{X_u}{X_v} = 0.$$

From the second of these equations we have also

$$[4'] \qquad \frac{\partial \log \rho}{\partial u} = - \frac{\partial}{\partial u} \log \frac{X_u}{X_v}.$$

The necessary and sufficient condition that

$$(2') \qquad \pi - m = 0$$

shall admit $\overline{U}'f$ may be written

$$\rho \frac{m_u}{m} - \rho \frac{X_u}{X_v} \frac{m_v}{m} + \rho \frac{X_v X_{uv} - X_u X_{vv}}{X_v^2} + \rho_u = 0;$$

using the relation [4′] we have

$$[5] \qquad \frac{m_u}{m} - \frac{X_u}{X_v}\frac{m_v}{m} + \frac{X_v X_{uv} - X_u X_{vv}}{X_v^2} + \frac{X_u X_{uv} - X_v X_{uu}}{X_u X_v} = 0.$$

It is now easy to show that this condition [5] is satisfied whenever the assumed necessary condition of §3 is satisfied.

Multiply [5] by X_v and rearrange the terms, and there results

$$[5'] \qquad X_v\left(\frac{m_u}{m} + \frac{X_{uv}}{X_v} - \frac{X_{uu}}{X_u}\right) - X_u\left(\frac{m_v}{m} + \frac{X_{vv}}{X_v} - \frac{X_{uv}}{X_u}\right) = 0.$$

Equation [5′] may be given the determinant form:

$$[5''] \qquad \begin{vmatrix} \dfrac{\partial}{\partial u}\left[\log\left(m\dfrac{X_v}{X_u}\right)\right] & \dfrac{\partial}{\partial v}\left[\log\left(m\dfrac{X_v}{X_u}\right)\right] \\ \dfrac{\partial}{\partial u}X & \dfrac{\partial}{\partial v}X \end{vmatrix} = 0.$$

But [5″] is always satisfied when X and I are functions of the same harmonic function $U + V$. We have therefore the

THEOREM. *A necessary and sufficient condition that an ordinary differential equation of the first order shall admit a one-parameter conformal group is that a conformal invariant of the third order and its first derived invariant of the fourth order shall be functions of the same harmonic function.*

5. Interpretation of the Invariants. We may write the invariants

$$X = \frac{1}{(ma)^{\frac{1}{2}}}\frac{\partial}{\partial u}\log\frac{a}{m}, \qquad Y = \left(\frac{m}{a}\right)\frac{\partial}{\partial v}\log am$$

in the original coordinates x and y. We shall for that purpose adopt the notation

$$\tau = \text{arc tan } a,$$

$$\Delta\tau = \frac{\partial^2\tau}{\partial x^2} + \frac{\partial^2\tau}{\partial y^2},$$

and denote partial differentiation where convenient by subscripts.

We furthermore find it convenient to multiply X and Y by $(-2i)^{\frac{1}{2}}$.

We have then:

$$X' = \frac{1}{[\Delta\tau(1+a^2)]^{\frac{1}{2}}}\left[\frac{\Delta\tau_x + a\Delta\tau_y}{\Delta\tau} + \frac{2(a_y - aa_x)}{1+a^2} \right.$$
$$\left. + i\left(\frac{a\Delta\tau_x - \Delta\tau_y}{\Delta\tau} + \frac{2(aa_y + a_x)}{1+a^2}\right)\right],$$

$$Y' = \frac{1}{[\Delta\tau(1+a^2)]^{\frac{1}{2}}}\left[\frac{\Delta\tau_x + a\Delta\tau_y}{\Delta\tau} + \frac{2(a_y - aa_x)}{1+a^2} \right.$$
$$\left. - i\left(\frac{a\Delta\tau_x - \Delta\tau_y}{\Delta\tau} + \frac{2(aa_y + a_x)}{1+a^2}\right)\right].$$

Adding X' and Y' and subtracting, in order to obtain real forms, we have finally the two independent invariants of the third order:

$$\bar{X} = \frac{1}{[\Delta\tau(1+a^2)^{\frac{1}{2}}]}\left[\frac{a\Delta\tau_y + \Delta\tau_x}{\Delta\tau} - \frac{2(a_y - aa_x)}{1+a^2}\right],$$
$$\bar{Y} = \frac{1}{[\Delta\tau(1+a^2)]^{\frac{1}{2}}}\left[\frac{a\Delta\tau_x - \Delta\tau_y}{\Delta\tau} + \frac{2(aa_y + a_x)}{1+a^2}\right].$$

Denoting the curvature of the integral curves of the original differential equation by κ, we have:

$$\kappa = \frac{a_x + aa_y}{[1+a^2]^{\frac{3}{2}}}.$$

Similarly writing the curvature of the orthogonal trajectories,

$$\bar{\kappa} = \frac{aa_x - a_y}{(1+a^2)^{\frac{3}{2}}},$$

we may then write $\Delta\tau = \frac{d\kappa}{ds} + \frac{d\bar{\kappa}}{d\bar{s}}$ where s and $\bar{s}$ denote arcs on the integral curve and the orthogonal trajectory respectively. With this understanding we may write the invariants $\bar{X}$ and $\bar{Y}$ as functions of the curvatures κ and $\bar{\kappa}$ and their first and second derivatives with respect to the arcs.

$$\bar{X} = \frac{1}{(\kappa_s + \bar{\kappa}_{\bar{s}})^{\frac{1}{2}}}\left[\frac{\kappa_{ss} + \bar{\kappa}_{s\bar{s}}}{\kappa_s + \bar{\kappa}_{\bar{s}}} - \frac{2\bar{\kappa}}{(\kappa_s + \bar{\kappa}_{\bar{s}})^{\frac{1}{2}}}\right],$$

$$\bar{Y} = \frac{1}{(\kappa_s + \bar{\kappa}_{\bar{s}})^{\frac{1}{2}}}\left[\frac{\kappa_{s\bar{s}} + \bar{\kappa}_{\bar{s}\bar{s}}}{\kappa_s + \bar{\kappa}_{\bar{s}}} + \frac{2\kappa}{(\kappa_s + \bar{\kappa}_{\bar{s}})^{\frac{1}{2}}}\right].$$

The fourth order invariants may be given similar form. In terms of the original variables x, y, a, a differential parameter is

$$\frac{a\Omega_y + \Omega_x}{\Omega_y - a\Omega_x},$$

where Ω is *any* invariant. Hence we may write as the two fourth order invariants,

$$\bar I = \frac{a\bar X_y + \bar X_x}{\bar X_y - a\bar X_x}, \qquad \bar J = \frac{a\bar Y_y + \bar Y_x}{\bar Y_x + a\bar Y_y}.$$

Expressing these invariants in terms of the curvatures κ and $\bar\kappa$ and their derivatives we have,

$$\bar I = P/Q,\ \bar J = R/S,$$

where

$$P = 2\left[\frac{\partial^2}{\partial s^2}\log(\kappa_s + \bar\kappa_{\bar s}) - \frac{1}{2}\left\{\frac{\partial}{\partial s}\log(\kappa_s + \bar\kappa_{\bar s})\right\}^2 + \frac{2\bar\kappa(\kappa_{ss} + \bar\kappa_{\bar s\bar s})}{(\kappa_s + \bar\kappa_{\bar s})^{\frac{3}{2}}} - \frac{2\bar\kappa_s}{(\kappa_s + \bar\kappa_{\bar s})^{\frac{1}{2}}}\right]$$
$$+\left(\frac{1-a^2}{a}\right)\left[\frac{\partial^2}{\partial s\partial\bar s}\log(\kappa_s + \bar\kappa_{\bar s}) - \frac{1}{2}\left\{\frac{\partial}{\partial s}\log(\kappa_s + \bar\kappa_{\bar s})\frac{\partial}{\partial\bar s}\log(\kappa_s + \bar\kappa_{\bar s})\right\}\right.$$
$$\left.+\frac{2\bar\kappa(\kappa_{s\bar s} + \bar\kappa_{\bar s\bar s})}{(\kappa_s + \bar\kappa_{\bar s})^{\frac{3}{2}}} - \frac{2\bar\kappa_{\bar s}}{(\kappa_s + \bar\kappa_{\bar s})^{\frac{1}{2}}}\right]$$

$$Q = \left(\frac{1-a^2}{a}\right)\left[\frac{\partial^2}{\partial s^2}\log(\kappa_s + \bar\kappa_{\bar s}) - \frac{1}{2}\left\{\frac{\partial}{\partial s}\log(\kappa_s + \bar\kappa_{\bar s})\right\}^2 + \frac{2\bar\kappa(\kappa_{ss} + \bar\kappa_{\bar s\bar s})}{(\kappa_s + \bar\kappa_{\bar s})^{\frac{3}{2}}}\right.$$
$$\left.-\frac{2\bar\kappa_s}{(\kappa_s + \bar\kappa_{\bar s})^{\frac{1}{2}}}\right] - 2\left[\frac{\partial^2}{\partial s\partial\bar s}\log(\kappa_s + \bar\kappa_{\bar s}) - \frac{1}{2}\left\{\frac{\partial}{\partial s}\log(\kappa_s + \bar\kappa_{\bar s})\frac{\partial}{\partial\bar s}\log(\kappa_s + \bar\kappa_{\bar s})\right\}\right.$$
$$\left.+\frac{2\bar\kappa(\kappa_{s\bar s} + \kappa_{\bar s\bar s})}{(\kappa_s + \kappa_{\bar s})^{\frac{3}{2}}} - \frac{2\bar\kappa_{\bar s}}{(\kappa_s + \bar\kappa_{\bar s})^{\frac{1}{2}}}\right],$$

$$R = 2\left[\frac{\partial^2}{\partial s\partial\bar s}\log\ (\kappa_s + \bar\kappa_{\bar s}) - \frac{1}{2}\left\{\frac{\partial}{\partial s}\log\ (\kappa_s + \bar\kappa_{\bar s})\frac{\partial}{\partial\bar s}\log(\kappa_s + \bar\kappa_{\bar s})\right\} - \frac{2\kappa(\kappa_{ss} + \bar\kappa_{\bar s\bar s})}{(\kappa_s + \bar\kappa_{\bar s})^{\frac{3}{2}}}\right.$$
$$\left.-\frac{2\kappa_s}{(\kappa_s + \bar\kappa_{\bar s})^{\frac{1}{2}}}\right] + \left(\frac{1-a^2}{a}\right)\left[\frac{\partial^2}{\partial\bar s^2}\log(\kappa_s + \bar\kappa_{\bar s}) - \frac{1}{2}\left\{\frac{\partial}{\partial\bar s}\log(\kappa_s + \bar\kappa_{\bar s})\right\}^2\right.$$
$$\left.-\frac{2\kappa(\kappa_{s\bar s} + \bar\kappa_{\bar s\bar s})}{(\kappa_s + \bar\kappa_{\bar s})^{\frac{2}{2}}} + \frac{2\kappa_{\bar s}}{(\kappa_s + \kappa_{\bar s})^{\frac{1}{2}}}\right],$$

$$S=\left(\frac{1-a^2}{a}\right)\left[\frac{\partial^2}{\partial s\partial\bar{s}}\log(\kappa^{s}+\bar{\kappa}_{\bar{s}})-\frac{1}{2}\left\{\frac{\partial}{\partial\bar{s}}\log(\kappa_s+\bar{\kappa}_{\bar{s}})\frac{\partial}{\partial s}\log(\kappa_s+\bar{\kappa}_{\bar{s}})\right\}\right.$$
$$\left.-\frac{2\kappa(\kappa_{ss}+\bar{\kappa}_{\bar{s}s})}{(\kappa_s+\bar{\kappa}_{\bar{s}})^{\frac{3}{2}}}+\frac{2\kappa_s}{(\kappa_s+\bar{\kappa}_{\bar{s}})^{\frac{1}{2}}}\right]-2\left[\frac{\partial^2}{\partial\bar{s}^2}\log(\kappa_s+\bar{\kappa}_{\bar{s}})-\frac{1}{2}\left\{\frac{\partial}{\partial\bar{s}}\log(\kappa_s+\bar{\kappa}_{\bar{s}})\right\}^2\right.$$
$$\left.-\frac{2\kappa(\kappa_{s\bar{s}}+\bar{\kappa}_{\bar{s}\bar{s}})}{(\kappa_s+\bar{\kappa}_{\bar{s}})^{\frac{3}{2}}}+\frac{2\kappa_s}{(\kappa_s+\bar{\kappa}_{\bar{s}})^{\frac{1}{2}}}\right].$$

6. Conclusion.

It is possible to write down the form that the function a must take in the differential equation

$$\frac{dy}{dx}=a(x,y) \tag{1}$$

in order that the equation shall admit a conformal group. Since the path curves $\omega=c$, of such a group must individually cut each integral curve under a constant angle along that curve we have at once

$$\frac{\dfrac{dy}{dx}+\dfrac{\omega_x}{\omega_y}}{1-\dfrac{dy}{dx}\dfrac{\omega_x}{\omega_y}}=F(\omega), \tag*{[6]}$$

where ω is a solution of Laplace's equation,

$$\omega_{xx}+\omega_{yy}=0.$$

Solving [6] for $\frac{dy}{dx}$, we obtain

$$\frac{dy}{dx}=\frac{-\omega_x+F(\omega)\omega_y}{\omega_y+F(\omega)\omega_x}. \tag*{[7]}$$

The corresponding infinitesimal conformal transformation is

$$Uf:\quad\frac{\omega_y}{\omega_x^2+\omega_y^2}\frac{\partial f}{\partial x}-\frac{\omega_x}{\omega_x^2+\omega_y^2}\frac{\partial f}{\partial y}.$$

It will be found that whenever a has the form in equation [7] that X, Y, I, and J are functions of ω, or in other words the form [7] is necessary and sufficient.

YALE UNIVERSITY,
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